Name________________________________

Activity - Definite Integrals

Part 1. Suppose that the following information is known about a function f: the graph of its derivative, $y = f'(x)$, is given in Figure. Further, assume that f' is piecewise linear (as pictured) and that for $x \le 0$ and $x \ge 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

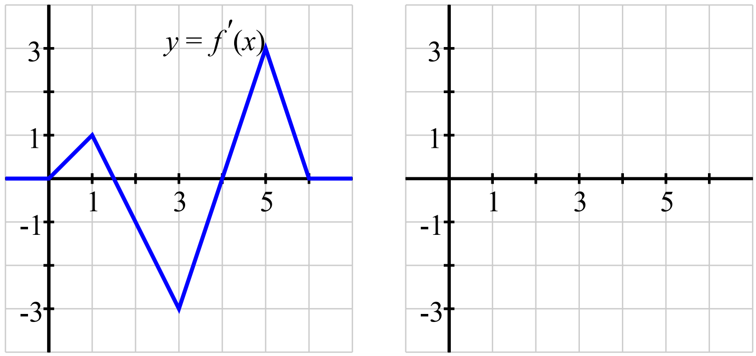

Figure: At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.

(a) On what interval(s) is f an increasing function? On what intervals is f decreasing?

(b) On what interval(s) is f concave up? concave down?

(c) At what point(s) does f have a relative minimum? a relative maximum?

(d) The Net Change Theorem tells us that

$$f(1) - f(0) = \int_0^1 f'(x)\, dx.$$

Recall that the definite integral from a to b is the area under the curev from $x = a$ to $x = b$. Use the graph to find this area. Using the known area and the given value of $f(0)$, find the exact value of $f(1)$.

(e) Using the same technique as part (d) find the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.

(f) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of f for $x < 0$ and $x > 6$.

(g) What happens if we change one key piece of information: in particular, say that g is an antiderivative of f' and $g(0) = 0$. How (if at all) would your answers to the preceding questions cnange? Sketch a graph of g on the same axes as the graph of f you constructed in (f).

Part 2.

Suppose that the following information is known about the functions f, g, x^2, and x^3:

- $\int_0^2 f(x)\,dx = -3$; $\int_2^5 f(x)\,dx = 2$
- $\int_0^2 g(x)\,dx = 4$; $\int_2^5 g(x)\,dx = -1$
- $\int_0^2 x^2\,dx = \frac{8}{3}$; $\int_2^5 x^2\,dx = \frac{117}{3}$
- $\int_0^2 x^3\,dx = 4$; $\int_2^5 x^3\,dx = \frac{609}{4}$

Use the provided information and the rules for definite integrals to evaluate each of the following definite integrals.

(a) $\int_5^2 f(x)\,dx$

(b) $\int_0^5 g(x)\,dx$

(c) $\int_0^5 (f(x) + g(x))\,dx$

(d) $\int_2^5 (3x^2 - 4x^3)\,dx$

(e) $\int_5^0 (2x^3 - 7g(x))\,dx$

Part 3.

1. In exercises a-d, use any symmetry and use the identity

$$\int_{-A}^{A} f(x)dx = \int_{-A}^{0} f(x)dx + \int_{0}^{A} f(x)dx$$ to compute the integrals.

a) $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2}dt$ (*Hint*: $\sin(-t) = -\sin(t)$)

b) $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t}dt$

c) $\int_1^3 (2-x)dx$ (*Hint:* Look at the graph of *f*.)

d) $\int_2^4 (x-3)^3\,dx$ (*Hint:* Look at the graph of *f*.)

e) Graph of $\sin(2\pi x)$ and then:

1. Explain why $\int_0^1 \sin(2\pi t)dt = 0$.
2. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)dt = 0$ for any value of a.

Part 4.

Suppose that $v(t) = \sqrt{4 - (t-2)^2}$ tells us the instantaneous velocity of a moving object on the interval $0 \leq t \leq 4$, where t is measured in minutes and v is measured in meters per minute.

(a) Sketch an accurate graph of $y = v(t)$. What kind of curve is $y = \sqrt{4 - (t-2)^2}$?

(b) Evaluate $\int_0^4 v(t)\, dt$ exactly.

(c) In terms of the physical problem of the moving object with velocity $v(t)$, what is the meaning of $\int_0^4 v(t)\, dt$? Include units on your answer.

(d) The average value of a function is found by computing

$$V_{ave} = \frac{1}{b-a} \int_a^b v(t) dt$$

Determine the exact **average value** of $v(t)$ on $[0, 4]$. Include units on your answer.

(e) Sketch a rectangle whose base is the line segment from $t = 0$ to $t = 4$ on the t-axis such that the rectangle's area is equal to the value of $\int_0^4 v(t)\, dt$. What is the rectangle's exact height?

(f) How can you use the average value you found in (d) to compute the total distance traveled by the moving object over $[0, 4]$?

Part 5.

Use your knowledge of derivatives of basic functions to complete the table of antiderivatives. For each entry, your task is to find a function F whose derivative is the given function f. When finished, use the FTC and the results in the table to evaluate the three given definite integrals below.

given function, $f(x)$	antiderivative, $F(x)$
k, (k is constant)	
x^n, $n \neq -1$	
$\frac{1}{x}$, $x > 0$	
$\sin(x)$	
$\cos(x)$	
$\sec(x)\tan(x)$	
$\csc(x)\cot(x)$	
$\sec^2(x)$	
$\csc^2(x)$	
e^x	
a^x $(a > 1)$	
$\frac{1}{1+x^2}$	
$\frac{1}{\sqrt{1-x^2}}$	

Table: Familiar basic functions and their antiderivatives.

(a) $\int_0^1 \left(x^3 - x - e^x + 2\right)\, dx$

(b) $\int_0^{\pi/3} (2\sin(t) - 4\cos(t) + \sec^2(t) - \pi)\, dt$

(c) $\int_0^1 (\sqrt{x} - x^2)\, dx$