Proof of the Ratio Test

The Ratio Test has three parts, (a), (b), and (c), and each part requires a separate proof.

(a) $L<1 \Rightarrow$ the series is absolutely convergent. The basic pattern of the proof for this part is to show that the given series is, term–by–term, less than a convergent geometric series. Then we conclude by the Comparison Test that the given series converges.

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$. Then there is a number r between L and 1 so that the ratios $\left| \frac{a_{n+1}}{a_n} \right|$ are eventually (for all $n >$ some N) less than r :

for all $n > N$, $\left| \frac{a_{n+1}}{a_n} \right| < r < 1$.

Then $|a_{N+1}| < r|a_N|$, $|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$, $|a_{N+3}| < r|a_{N+2}| < r^3|a_N|$, and, in general, $|a_{N+k}| < r^k|a_N|$.

So $|a_N| + |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \ldots$

$$< |a_N| + r|a_N| + r^2|a_N| + r^3|a_N| + r^4|a_N| + \ldots$$

$$= |a_N| \cdot \{ 1 + r + r^2 + r^3 + r^4 + \ldots \}$$

$= |a_N| \cdot \frac{1}{1-r}$, since the powers of r form a convergent geometric series,

and the series $\sum_{n=N}^{\infty} |a_n|$ is convergent by the Comparison Test.

Finally, we can conclude that the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$ is convergent, since it is the sum of two convergent series. $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) $L>1 \Rightarrow$ the series is divergent. The basic idea in this part is to show that the terms of the given series do not approach 0. Then we can conclude by the N^{th} Term Test that the given series diverges.

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$. Then the ratios $\left| \frac{a_{n+1}}{a_n} \right|$ are eventually (for all $n >$ some N) larger than 1 so $|a_{N+1}| > |a_N|$, $|a_{N+2}| > |a_{N+1}| > |a_N|$, $|a_{N+3}| > |a_{N+2}| > |a_N|$, and, for all $k > N$, $|a_k| > |a_N|$.

Thus $\lim_{n\to\infty} |a_n| \geq |a_N| \neq 0$ and $\lim_{n\to\infty} a_n \neq 0$, so by the N^{th} Term Test for Divergence (Section 10.2) we can conclude that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Part (c) proof was in the video.