Name______________________________

Activity – The Dot Product

Part 1. Definition and Algebraic Properties of the Dot Product

Definition. Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors whose component forms are $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$. The **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v}\cdot\mathbf{w}=\langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle=v_1w_1+v_2w_2$$

a) Do you get a vector or a scalar when you compute the dot product of two vectors?
b) Does $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$?

c) Find $\mathbf{v}\cdot\mathbf{v}$. Find $\|\mathbf{v}\|$. How are these two quantities related?

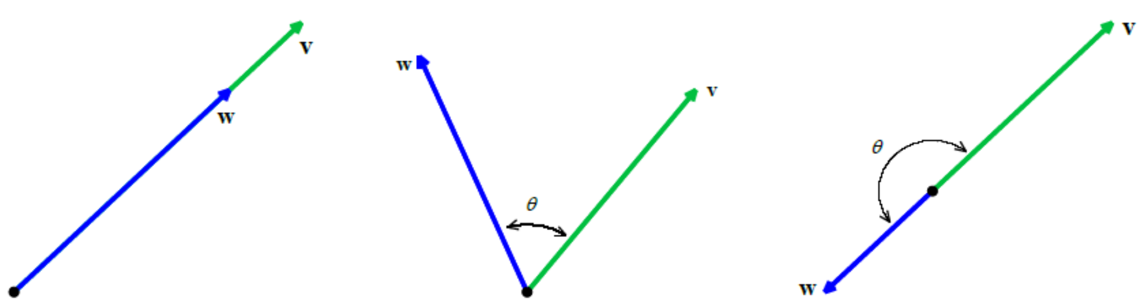
d) $\mathbf{v}=\langle -2,1\rangle$ and $\mathbf{w}=\langle 3,-3\rangle$. Find $\mathbf{v}\cdot\mathbf{w}$.

Theorem 9.5. Properties of the Dot Product:

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$.
- **Distributive Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$.
- **Scalar Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, and scalars k, $(k\mathbf{v})\cdot\mathbf{w}=k(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(k\mathbf{w})$.
- **Relation to Magnitude:** For all vectors $\mathbf{v}$, $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$.

Part 2. Geometric Properties of the Dot Product

Let θ be the angle between two vectors v and w. We choose to define $0\le\theta\le\pi$. Find θ for the vectors shown below.



Theorem 9.6. Geometric Interpretation of the Dot Product: If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors then $\mathbf{v}\cdot\mathbf{w}=\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Part 3. Determining the Angle Between Two Vectors.

Using $\mathbf{v}\cdot\mathbf{w}=\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, solve for θ.

Theorem 9.7. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\theta = \boxed{}$$

Find the angle between the following pairs of vectors:

1. $\mathbf{v}=\langle 3,-3\sqrt{3}\rangle$ and $\mathbf{w}=\langle -\sqrt{3},1\rangle$

2. $\mathbf{v}=\langle 2,2\rangle$ and $\mathbf{w}=\langle 5,-5\rangle$

3. $\mathbf{v}=\langle 3,-4\rangle$ and $\mathbf{w}=\langle 2,1\rangle$

If the angle between two vectors is $\frac{\pi}{2}$ or $90°$, we call the vectors ______________________

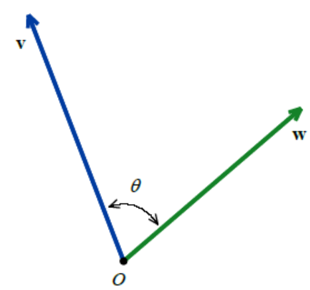
and we write ________________.

If **v** and **w** are orthogonal, what would $\mathbf{v}\cdot\mathbf{w}$ equal?

Part 4. Orthogonal Projection

1. Draw a perpendicular line from the tip of **v** to **w**.
2. Label the point of intersection with **w** R.
3. Make a new vector $\mathbf{p}=\overrightarrow{OR}$.

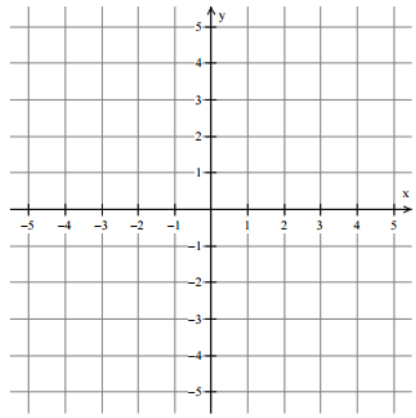
This new vector **p** is called the projection of **v** onto **w**.



Theorem 9.9. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then the **orthogonal projection of** $\mathbf{v}$ **onto** $\mathbf{w}$, denoted $\text{proj}_{\mathbf{w}}(\mathbf{v})$, is given by

$$\text{proj}_{\mathbf{w}}(\mathbf{v})=\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

Given vectors $\mathbf{v} = \langle -2, 4 \rangle$ and $\mathbf{w} = \langle 3, 3 \rangle$, find $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v})$.



Plot **v**, **w** and **p** in standard position on the graph.

Part 5. Work

Theorem 9.10. Work as a Dot Product: Suppose a constant force $\mathbf{F}$ is applied to move an object along the vector $\overrightarrow{PQ}$, from P to Q. The work W done by $\mathbf{F}$ is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos(\theta),$$

where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$.

Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon?